AD-A239 579

(2)

AD

TECHNICAL REPORT ARCCB-TR-91024

# AN EXTENSION OF MESH EQUIDISTRIBUTION TO TIME-DEPENDENT PARTIAL DIFFERENTIAL EQUATIONS

J. M. COYLE



JULY 1991

US ARMY ARMAMENT RESEARCH, DEVELOPMENT AND ENGINEERING CENTER
CLOSE COMBAT ARMAMENTS CENTER
BENÉT LABORATORIES
WATERVLIET, N.Y. 12189-4050

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

## DISCLAIMER

The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.

The use of trade name(s) and/or manufacturer(s) does not constitute an official indorsement or approval.

## DESTRUCTION NOTICE

For classified documents, follow the procedures in DoD 5200.22-M, Industrial Security Manual, Section II-19 or DoD 5200.1-R, Information Security Program Regulation, Chapter IX.

For unclassified, limited documents, destroy by any method that will prevent disclosure of contents or reconstruction of the document.

For unclassified, unlimited documents, destroy when the report is no longer needed. Do not return it to the originator.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

# REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

| | |
|---|---|
| 1. REPORT NUMBER<br>ARCCB-TR-91024 | 2. GOVT ACCESSION NO. |
| 3. RECIPIENT'S CATALOG NUMBER | |
| 4. TITLE (and Subtitle)<br>AN EXTENSION OF MESH EQUIDISTRIBUTION TO TIME-DEPENDENT PARTIAL DIFFERENTIAL EQUATIONS | 5. TYPE OF REPORT & PERIOD COVERED<br>Final |
| | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br>J.M. Coyle | 8. CONTRACT OR GRANT NUMBER(s) |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>U.S. Army ARDEC<br>Benet Laboratories, SMCAR-CCB-TL<br>Watervliet, NY 12189-4050 | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br>AMCMS No. 6111.02.H610.011<br>PRON No. 1A04Z0CANMSC |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>U.S. Army ARDEC<br>Close Combat Armaments Center<br>Picatinny Arsenal, NJ 07806-5000 | 12. REPORT DATE<br>July 1991 |
| | 13. NUMBER OF PAGES<br>12 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | 15. SECURITY CLASS. (of this report)<br>UNCLASSIFIED |
| | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Presented at the Eighth Army Conference on Applied Mathematics and Computing, Cornell University, Ithaca, NY, 19-22 June 1990.
Published in Proceedings of the Conference.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Partial Differential Equations
Finite Elements
Mesh Moving
Equidistribution

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Mesh equidistribution for static approximation problems is extended to a domain that admits time dependency. Instead of some quantity being equidistributed over a static mesh, the change in this quantity is equidistributed over a dynamic, moving mesh. Applications are for utilization in numerical methods to solve time-dependent partial differential equations. This mesh moving scheme is incorporated into a finite element code which already refines adaptively. Comparisons are made for stationary and moving meshes.

## TABLE OF CONTENTS



## LIST OF ILLUSTRATIONS

## INTRODUCTION

An equidistributed mesh in one space dimension is a partition of a given domain into subintervals such that some given quantity is uniform over each subinterval. More specifically, given an interval (a,b) and a positive weight function w(x) defined on (a,b), then an equidistributed mesh is a partition

$$\{a = x_0 < x_1 < x_2 < \ldots < x_{M-1} < x_M = b\}$$

such that

$$\int_{x_{j-1}}^{x_j} w(x)dx = \text{constant} = \frac{1}{M}\int_a^b w(x)dx \quad , \quad j = 1,2,\ldots,M \tag{1}$$

The usual application of such a mesh is for approximating functional relationships to a certain accuracy with a minimum number of mesh points by choosing w(x) appropriately (ref 1). Equidistribution strategies have also been used in numerical methods for solving two-point boundary value problems (refs 2,3). This is because it has been shown (refs 4,5) that the task of selecting a mesh to minimize the discretization error is asymptotically equivalent to equidistributing the local discretization error.

The successes in the above fields of functional approximation and numerical ordinary differential equations have led some investigators to consider the use of equidistribution strategies for generating moving meshes in the field of numerical partial differential equations (PDEs) (refs 6-8). The general framework is to simply reconsider Eq. (1) with a time dependency. That is to say, the problem is now to determine a dynamic mesh

$$\{a = x_0 < x_1(t) < x_2(t) < \ldots < x_{M-1}(t) < x_M = b\}$$

at time t so that

$$\int_{x_{j-1}(t)}^{x_j(t)} w(x,t)dx = c(t) = \frac{1}{M}\int_a^b w(x,t)dx \quad , \quad j = 1,2,\ldots,M \tag{2}$$

where the positive weight function $w(x,t)$ is usually chosen to be a function of the solution of the underlying PDE. For example, w has been chosen to be proportional to the solution's gradient, curvature, and local discretization error.

When applying Eq. (2) in some numerical scheme, most investigators move a fixed number of points to follow and resolve local nonuniformities in the solution. In order to guarantee a certain accuracy, they must be sure that this fixed number is large enough to approximate the solution throughout the entire spatial domain for the entire temporal "life" of the solution. Some see this as a limitation since the correct number of fixed points necessary is not generally known a priori.

Also, this moving mesh is not operating in a vacuum. It is being used in conjunction with some numerical solution procedure. Since the accuracy of most such procedures can depend on the shape of the space-time grid, sometimes the equidistribution law can be too dynamic and deform the grid enough to introduce a new, even larger source of error. This can happen even if the equidistribution law does not demand much moving of its own accord. If a nonequidistributed mesh is used as the initial mesh, then the grid can deform drastically as the moving mesh tries to relocate to the proper equidistributed positions. In order to avoid these difficulties, some investigators have abandoned moving altogether and developed local refinement methods (ref 3).

A local refinement method is a procedure where uniform fine grids are added to coarse grids in regions where the solution is not adequately resolved. Although they can guarantee a solution to a prescribed accuracy, they can be costly, as they involve recomputing the solution, and they are not as good as moving mesh methods at reducing dispersive errors in the vicinity of wavefronts.

The choice here has been to combine local refinement with mesh moving based on equidistribution. The purpose is twofold. First, the refinement procedure is incorporated to avoid any drastic deformation of the grid by the moving mesh as well as guarantee a prescribed accuracy. Second, the mesh moving is applied in order to obtain as accurate a solution as possible for any given discretization to delay the need for refinement for as long as possible and thus to reduce the costs involved.

Equation (2) as it stands, however, is not easily partnered with a refinement scheme. It is too dependent on mesh position and the number of extant mesh points. Hence, a refinement step can disrupt the nature of the equidistribution and cause a drastic change in the mesh dynamics similar to that caused by a "bad" initial mesh.

The attempt to overcome the difficulty reported here was to try to extend Eq. (2) in such a way that it worked with the refinement procedure rather than against it. It seemed that the dynamics of Eq. (2) was based on the static spatial nature of Eq. (1) and an extension was needed that incorporated more of the time dependency of the domain and solution process.

In the next section, this extension of Eq. (2) is presented as well as the algorithm for coupling the refinement and moving procedures. Then, in the following section, results on a series of test cases are presented for comparison. In the Discussion section, the characteristics of the extended equidistribution law are discussed in light of the results of the previous section. Finally, in the last section, some conclusions are presented.

## PROCEDURES

The basic principle behind this extension of Eq. (2) is to try to equidistribute temporal properties as well as spatial. To this end, consider a

typical time interval of interest (0,T) and a discretization

$$\{0 = t_0 < t_0 < t_1 < t_2 < \dots < t_{N-1} < t_N = T\}$$

of that interval. Then, given any time level $t_{n-1}$ and any mesh

$$\{a = x_0^{n-1} < x_1^{n-1} < x_2^{n-1} < \dots < x_M^{n-1} = b\}$$

at that time level, requires the new mesh

$$\{a = x_0^n < x_1^n < x_2^n < \dots < x_{M-1}^n < x_M^n = b\}$$

at the next time level $t_n$ to satisfy

$$\int_{x_{j-1}^n}^{x_j^n} w(x,t_n)dx = \int_{x_{j-1}^{n-1}}^{x_j^{n-1}} w(x,t_{n-1})dx + \frac{1}{M}\left\{\int_a^b w(x,t_n)dx - \int_a^b w(x,t_{n-1})dx\right\} ,$$

$$j = 1,2,\dots,M \tag{3}$$

Note that Eq. (3) is not an equidistribution law in the sense of Eqs. (1) and (2). No quantity is being held constant over any subinterval by the enforcement of Eq. (3). Rather, it is the change in the quantity w over each new subinterval

$$(x_{j-1}^n, x_j^n)$$

that is allowed to vary by a constant amount (which is proportional to the total change in w from $t_{n-1}$ to $t_n$) when compared to its values over the old subinterval

$$(x_{j-1}^{n-1}, x_j^{n-1})$$

In a sense then, it is the time change of this quantity that is being equidistributed.

Note also that the relationship between old and new meshes is not as dramatic as in Eq. (2). If

$$\{x_j^{n-1}\}_{j=0}^M$$

is a "bad" mesh, e.g., suppose it was readjusted by a refinement procedure, then Eq. (3) simply requires that

$$\{x_j^n\}_{j=0}^M$$

differs from

$$\{x_j^{n-1}\}_{j=0}^M$$

by a constant amount over each subinterval and does not require any drastic readjustment to a new equidistributed position.

In order to test the performance of Eq. (3) and its postulated properties, it was incorporated into a numerical PDE solver that already implemented an automatic refinement strategy. The overall solution algorithm is as follows:

1. Move mesh to next time level according to Eq. (3).
2. Solve PDE using finite elements in space and finite differences in time.
3. Estimate error that occurred in the solution process.
4. If the error is less than or equal to a prescribed tolerance and the time level is less than T, then go to step 1.
5. If the error is greater than the tolerance, refine in either space or time or both, then go to step 2.
6. If the time level is greater than or equal to T, then stop.

## RESULTS

The following PDE was solved numerically for all test cases:

$$u_t - u_x(1 + \frac{1}{10} u) = \frac{1}{200} u_{xx} \quad , \quad 0 < x < 1 \quad , t > 0$$

$$u(x,0) = \tanh 10(x-1) \quad , \quad 0 \leqslant x \leqslant 1$$

$$u(0,t) = \tanh 10(-1+t) \quad , \quad t \geqslant 0$$

$$u(1,t) = \tanh 10t \quad , \quad t \geqslant 0$$

The exact solution, $u(x,t) = \tanh 10(x-1+t)$, is simply a wavefront that moves through the spatial domain from right to left as time progresses. Optimally,

the mesh should try to follow the front as it moves across the interval (0,1).

For all cases, the $L_2$ norm of the error was prescribed to be less than a tolerance of 0.01, an initial uniform mesh with 11 points (M = 11) and an initial time step of 0.05 ($\Delta t = 0.05$) was input, and the solution process was allowed to proceed for 75 time steps.

**Case 1**

For this case, no movement was allowed--only refinement. Mesh trajectories are shown in Figure 1. At the end of 75 time steps, N = 37 and $\Delta t = 0.00218$.

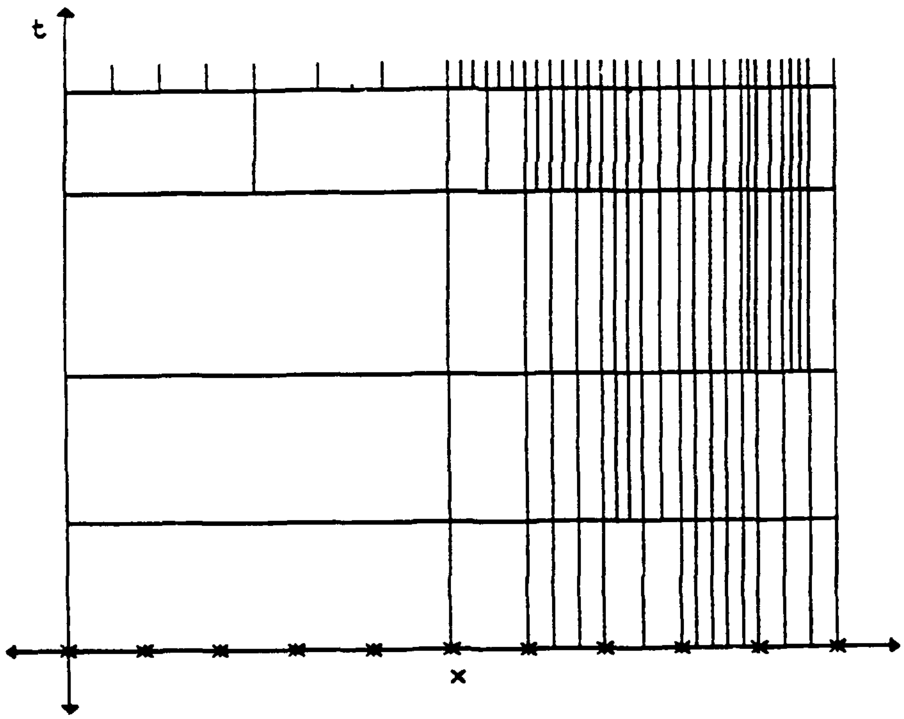


Figure 1. Mesh trajectories for case 1. Horizontal lines indicate a temporal refinement has occurred.

Mesh Trajectories for Case 1. Stars on x-axis indicate the mesh input before any moving or refining. Horizontal lines indicate a temporal refinement has occurred (actual values of $\Delta t$ are not shown).

## Case 2

For this case, movement was based on the first time derivative of the solution ($w(x,t) = |u_t(x,t)|$). Mesh trajectories are shown in Figure 2. At the end of 75 time steps, $N = 40$ and $\Delta t = 0.00579$.

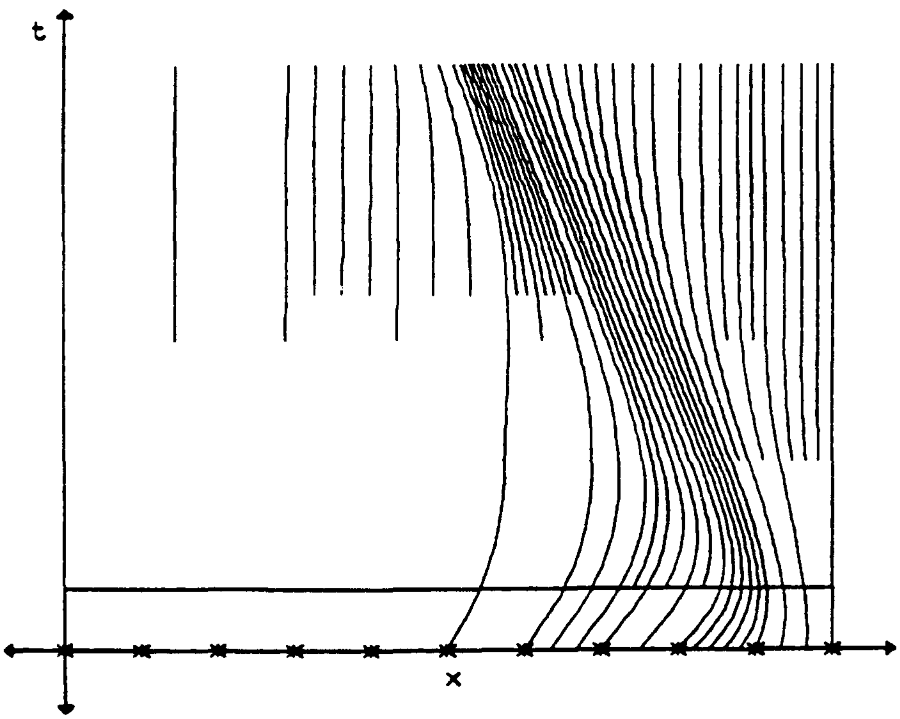


Figure 2. Mesh trajectories for case 2. Horizontal lines indicate a temporal refinement has occurred.

Mesh Trajectories for Case 2. Stars on x-axis indicate the mesh input before any moving or refining. Horizontal lines indicate a temporal refinement has occurred (actual values of $\Delta t$ are not shown).

## Case 3

For this case, movement was based on the second spatial derivative of the solution ($w(x,t) = |u_{xx}(x,t)|$). Mesh trajectories are shown in Figure 3. At the end of 75 time steps, N = 23 and $\Delta t$ = 0.00201.

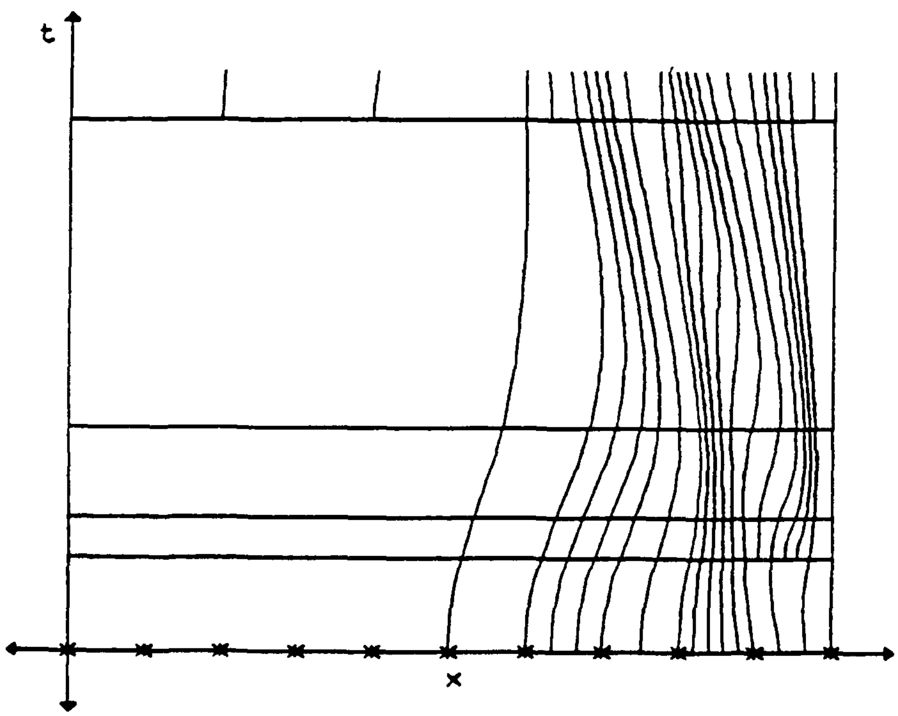


Figure 3. Mesh trajectories for case 3. Horizontal lines indicate a temporal refinement has occurred.

Mesh Trajectories for Case 3. Stars on x-axis indicate the mesh input before any moving or refining. Horizontal lines indicate a temporal refinement has occurred (actual values of $\Delta t$ are not shown).

## DISCUSSION

Overall, the results are very encouraging. The mesh trajectories flow smoothly with whatever solution characteristic the equidistribution law is based. This is true even when the initial mesh is unrelated to the equidistribution rule and when the refinement procedure alters the mesh (see Figures 2 and 3). This is exactly as desired and postulated.

Furthermore, it seems that mesh moving can decrease the amount of refinement necessary for a given problem as hoped. This is evident when comparing case 1 with cases 2 and 3.

In case 2, the level of temporal refinement is less than in case 1 for the same number of time steps and the same tolerance level. This is as expected since the temporal component of the error is proportional to a time derivative of the solution. Hence, movement based on equidistributing this error should reduce the temporal refinement necessary.

Similarly, in case 3, the level of spatial refinement is less than in case 1. Once again, this is as expected since the movement here is based on a quantity proportional to the spatial component of the error.

## CONCLUSIONS

Whether or not this new moving scheme will develop into a robust numerical procedure is still uncertain. There are still stability questions to be answered as well as some implementation difficulties not addressed here.

However, the results presented here give credence to the notion that mesh moving and refinement schemes can be successfully combined. Refinement procedures do not have to interfere with mesh movement, and mesh movement can be performed to reduce the levels of refinement necessary to solve a problem to a given tolerance.

## REFERENCES


1. C. deBoor, A Practical Guide to Splines, Springer-Verlag, New York, 1978.
2. U. Ascher, J. Christiansen, and R.D. Russell, "Collocation Software for Boundary-Value Codes," ACM Trans. Math. Software, Vol. 7, No. 2, June 1981, pp. 209-222.
3. M. Lentini and V. Pereyra, "An Adaptive Finite Difference Solver for Nonlinear Two-Point Boundary Problems With Mild Boundary Layers," SIAM J. Numer. Anal., Vol. 14, No. 1, March 1977, pp. 91-111.
4. M.J. Berger and J. Oliger, "Adaptive Mesh Refinement for Hyperbolic Partial Differential Equations," J. Comput. Phys., Vol. 53, 1984, pp. 484-512.
5. V. Pereyra and E.G. Sewell, "Mesh Selection for Discrete Solution of Boundary Problems in Ordinary Differential Equations," Numer. Math., Vol. 23, No. 3, 1975, pp. 261-268.
6. J.B. Bell and G.R. Shubin, "An Adaptive Grid Finite Difference Method for Conservation Laws," J. Comput. Phys., Vol. 52, 1983, pp. 569-591.
7. S.F. Davis and J.E. Flaherty, "An Adaptive Finite Element Method for Initial-Boundary Value Problems for Partial Differential Equations," SIAM J. Sci. Statist. Comput., Vol. 3, No. 1, 1982, pp. 6-27.
8. H.A. Dwyer, "A Discussion of Some Criteria For the Use of Adaptive Gridding," in: Adaptive Computational Methods for Partial Differential Equations, (I. Babuska, J. Chandra, and J.E. Flaherty, eds.), SIAM, Philadelphia, PA, 1983, pp. 111-122.

TECHNICAL REPORT INTERNAL DISTRIBUTION LIST

| | NO. OF COPIES |
|---|---|
| CHIEF, DEVELOPMENT ENGINEERING DIVISION | |
| ATTN: SMCAR-CCB-D | 1 |
| -DA | 1 |
| -DC | 1 |
| -DI | 1 |
| -DP | 1 |
| -DR | 1 |
| -DS (SYSTEMS) | 1 |
| CHIEF, ENGINEERING SUPPORT DIVISION | |
| ATTN: SMCAR-CCB-S | 1 |
| -SE | 1 |
| CHIEF, RESEARCH DIVISION | |
| ATTN: SMCAR-CCB-R | 2 |
| -RA | 1 |
| -RE | 1 |
| -RM | 1 |
| -RP | 1 |
| -RT | 1 |
| TECHNICAL LIBRARY<br>ATTN: SMCAR-CCB-TL | 5 |
| TECHNICAL PUBLICATIONS & EDITING SECTION<br>ATTN: SMCAR-CCB-TL | 3 |
| OPERATIONS DIRECTORATE<br>ATTN: SMCWV-ODP-P | 1 |
| DIRECTOR, PROCUREMENT DIRECTORATE<br>ATTN: SMCWV-PP | 1 |
| DIRECTOR, PRODUCT ASSURANCE DIRECTORATE<br>ATTN: SMCWV-QA | 1 |

<u>NOTE</u>: PLEASE NOTIFY DIRECTOR, BENET LABORATORIES, ATTN: SMCAR-CCB-TL, OF ANY ADDRESS CHANGES.

TECHNICAL REPORT EXTERNAL DISTRIBUTION LIST

| | NO. OF COPIES |
|---|---|
| ASST SEC OF THE ARMY<br>RESEARCH AND DEVELOPMENT<br>ATTN: DEPT FOR SCI AND TECH<br>THE PENTAGON<br>WASHINGTON, D.C. 20310-0103 | 1 |
| ADMINISTRATOR<br>DEFENSE TECHNICAL INFO CENTER<br>ATTN: DTIC-FDAC<br>CAMERON STATION<br>ALEXANDRIA, VA 22304-6145 | 12 |
| COMMANDER<br>US ARMY ARDEC<br>ATTN: SMCAR-AEE | 1 |
| SMCAR-AES, BLDG. 321 | 1 |
| SMCAR-AET-O, BLDG. 351N | 1 |
| SMCAR-CC | 1 |
| SMCAR-CCP-A | 1 |
| SMCAR-FSA | 1 |
| SMCAR-FSM-E | 1 |
| SMCAR-FSS-D, BLDG. 94 | 1 |
| SMCAR-IMI-I (STINFO) BLDG. 59<br>PICATINNY ARSENAL, NJ 07806-5000 | 2 |
| DIRECTOR<br>US ARMY BALLISTIC RESEARCH LABORATORY<br>ATTN: SLCBR-DD-T, BLDG. 305<br>ABERDEEN PROVING GROUND, MD 21005-5066 | 1 |
| DIRECTOR<br>US ARMY MATERIEL SYSTEMS ANALYSIS ACTV<br>ATTN: AMXSY-MP<br>ABERDEEN PROVING GROUND, MD 21005-5071 | 1 |
| COMMANDER<br>HQ, AMCCOM<br>ATTN: AMSMC-IMP-L<br>ROCK ISLAND, IL 61299-6000 | 1 |
| COMMANDER<br>ROCK ISLAND ARSENAL<br>ATTN: SMCRI-ENM<br>ROCK ISLAND, IL 61299-5000 | 1 |
| DIRECTOR<br>US ARMY INDUSTRIAL BASE ENGR ACTV<br>ATTN: AMXIB-P<br>ROCK ISLAND, IL 61299-7260 | 1 |
| COMMANDER<br>US ARMY TANK-AUTMV R&D COMMAND<br>ATTN: AMSTA-DDL (TECH LIB)<br>WARREN, MI 48397-5000 | 1 |
| COMMANDER<br>US MILITARY ACADEMY<br>ATTN: DEPARTMENT OF MECHANICS<br>WEST POINT, NY 10996-1792 | 1 |
| US ARMY MISSILE COMMAND<br>REDSTONE SCIENTIFIC INFO CTR<br>ATTN: DOCUMENTS SECT, BLDG. 4484<br>REDSTONE ARSENAL, AL 35898-5241 | 2 |
| COMMANDER<br>US ARMY FGN SCIENCE AND TECH CTR<br>ATTN: DRXST-SD<br>220 7TH STREET, N.E.<br>CHARLOTTESVILLE, VA 22901 | 1 |
| COMMANDER<br>US ARMY LABCOM<br>MATERIALS TECHNOLOGY LAB<br>ATTN: SLCMT-IML (TECH LIB)<br>WATERTOWN, MA 02172-0001 | 2 |

NOTE: PLEASE NOTIFY COMMANDER, ARMAMENT RESEARCH, DEVELOPMENT, AND ENGINEERING CENTER, US ARMY AMCCOM, ATTN: BENET LABORATORIES, SMCAR-CCB-TL, WATERVLIET, NY 12189-4050, OF ANY ADDRESS CHANGES.

TECHNICAL REPORT EXTERNAL DISTRIBUTION LIST (CONT'D)

| | NO. OF COPIES |
|---|---|
| COMMANDER<br>US ARMY LABCOM, ISA<br>ATTN: SLCIS-IM-TL<br>2800 POWDER MILL ROAD<br>ADELPHI, MD 20783-1145 | 1 |
| COMMANDER<br>US ARMY RESEARCH OFFICE<br>ATTN: CHIEF, IPO<br>P.O. BOX 12211<br>RESEARCH TRIANGLE PARK, NC 27709-2211 | 1 |
| DIRECTOR<br>US NAVAL RESEARCH LAB<br>ATTN: MATERIALS SCI & TECH DIVISION<br>CODE 26-27 (DOC LIB)<br>WASHINGTON, D.C. 20375 | 1<br>1 |
| DIRECTOR<br>US ARMY BALLISTIC RESEARCH LABORATORY<br>ATTN: SLCBR-IB-M (DR. BRUCE BURNS)<br>ABERDEEN PROVING GROUND, MD 21005-5066 | 1 |
| COMMANDER<br>AIR FORCE ARMAMENT LABORATORY<br>ATTN: AFATL/MN<br>EGLIN AFB, FL 32542-5434 | 1 |
| COMMANDER<br>AIR FORCE ARMAMENT LABORATORY<br>ATTN: AFATL/MNF<br>EGLIN AFB, FL 32542-5434 | 1 |
| MIAC/CINDAS<br>PURDUE UNIVERSITY<br>2595 YEAGER ROAD<br>WEST LAFAYETTE, IN 47905 | 1 |

NOTE: PLEASE NOTIFY COMMANDER, ARMAMENT RESEARCH, DEVELOPMENT, AND ENGINEERING CENTER, US ARMY AMCCOM, ATTN: BENET LABORATORIES, SMCAR-CCB-TL, WATERVLIET, NY 12189-4050, OF ANY ADDRESS CHANGES.